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Biorthogonal systems for $SU_4 \supset SU_2 \times SU_2$, $SU_n \supset SO_n$ and $Sp_4 \supset U_2$ and analytical inversion symmetry

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Abstract. Analytical inversion symmetry of the biorthogonal systems of $SU_4 \supset SU_2 \times SU_2$, $SU_n \supset SO_n$ and $Sp_4 \supset U_2$ bases for two-parametric (covariant and mixed tensor) irreducible representation is discovered. This symmetry relates the dual isoscalar factors or resubducing coefficients. It allowed us to invert, by means of a special analytical continuation procedure, the non-orthogonal isofactors of $SU_n \supset SO_n$ for couplings $(p\dot{0}) \times (\dot{0}q)$ to $(\lambda\dot{0}\mu)$ and $(p_1\dot{0}) \times (p_2 0)$ to $(\lambda\nu\dot{0})$, as well as the resubducing coefficients (transformation brackets) for expansion of the $SU_3 \supset SO_3$ Elliott basis states and $Sp_4 \supset U_2$ Smirnov and Tolstoy basis states in terms of the corresponding canonical basis states. New expressions for bilinear combinations of $SU_n \supset SO_n$ special isofactors are obtained. Expansion of SU_3 canonical basis states in terms of $SU_3 \supset SO_3$ Elliott states is found. Isofactors for coupling of two Elliott states are given.

1. Introduction

The concept of biorthogonal systems turned out to be rather an effective tool for consideration of different realisations of the non-canonical bases in the case of the multiple irreducible representations (irreps) of subgroups (see, e.g., Ališauskas 1978a, b, 1983c, 1984). It allows us to simplify considerably many operations of the Wigner-Racah algebra for the non-orthogonal basis states with multiplicity labels, especially for calculations needed in nuclear theory.

The biorthogonal system of non-canonical bases for the chain of the groups $G \supset H$ in the case of certain classes of irreps is formed by the two complete collections of the basis states $|\eta_a\rangle$ and $|\eta^b\rangle$ of the same irreps of G and H with

$$\langle \eta_a | \eta^b \rangle = \delta_{ab} \tag{1.1}$$

where a and b are the dual multiplicity labels. The arbitrary vector of this subspace may be expanded as follows:

$$|\zeta_c\rangle = \sum_b \langle \eta_b | \zeta_c \rangle | \eta^b \rangle = \sum_a \langle \eta^a | \zeta_c \rangle | \eta_a \rangle. \tag{1.2}$$

The analytical biorthogonal systems may be constructed by means of the dual resubducing coefficients (transformation brackets, cf Ališauskas 1978a, 1983c, 1984) taking into account that the isofactor of the coupling coefficient for the chain $G \supset H$ coincides with the resubducing coefficient (RC) for the chains

$$\begin{array}{ccc} G \times G & \supset & G \\ \cup & & \cup \\ H \times H & \supset & H. \end{array} \tag{1.3}$$

These dual resubducing coefficients (particularly the isofactors) are usually constructed in two complementary ways: the integral one (leading to some bilinear combinations of the orthonormal RC) and the differential one (associated with the solution of a discrete boundary value problem for RC). In general the resubducing coefficients are considered for the chains

$$\begin{array}{ccc} \mathrm{G} \supset \mathrm{H} & & \bar{\mathrm{G}} \supset \bar{\mathrm{H}} \\ \cup \quad \cup & \text{or} & \cup \quad \cup \\ \mathrm{G}' \supset \mathrm{H}' & & \mathrm{G} \supset \mathrm{H} \end{array} \tag{1.4}$$

where for the chosen classes of irreps only the restriction G ⊃ H is non-multiplicity-free. Let us denote here by λ, μ, j, m, Λ and M the irreps of G, H, G′, H′, $\bar{\mathrm{G}}$ and $\bar{\mathrm{H}}$, respectively, and by ω the multiplicity label of the orthogonal states of μ in λ.

The bilinear combinations of RC

$$\langle \lambda_{a'} \mu m \,|\, \lambda j m \rangle = \sum_{\omega} \langle \lambda \tilde{j} \tilde{m} \,|\, \lambda \omega \mu \tilde{m} \rangle \langle \lambda \omega \mu m \,|\, \lambda j m \rangle \tag{1.5a}$$

or

$$\langle \Lambda \lambda_{\bar{a}} \mu \,|\, \Lambda M \mu \rangle = \sum_{\omega} \langle \tilde{\Lambda} \tilde{M} \mu \,|\, \tilde{\Lambda} \lambda \omega \mu \rangle \langle \Lambda \lambda \omega \mu \,|\, \Lambda M \mu \rangle \tag{1.5b}$$

with one-to-one correspondence between a' and $\tilde{j}\tilde{m}$ or between $\bar{a}$ and $\tilde{\Lambda}\tilde{M}$ appear in the case of the non-canonical bases of the integral (projected or polynomial) type. (The explicit form of ω here is unnecessary and unimportant.) The parameters of irreps $\tilde{j}$, $\tilde{m}$ or $\tilde{\Lambda}$, $\tilde{M}$ in the first factors on RHS of (1.5a) and (1.5b) are chosen as being linearly dependent on λ and μ in such a way that the non-orthonormal basis formed with the help of these weight coefficients is complete when the possible superfluous vectors are eliminated with the additional inequalities. The overlaps of the non-orthonormal states may be expressed as particular cases of (1.5a) or (1.5b) (i.e. the bilinear combinations of the weight coefficients). The bilinear combinations of the weight coefficients of the different types, for example,

$$\sum_{\omega} \langle \tilde{\Lambda} \tilde{M} \mu \,|\, \tilde{\Lambda} \lambda \omega \mu \rangle \langle \lambda \omega \mu \tilde{m} \,|\, \lambda \tilde{j} \tilde{m} \rangle \tag{1.6}$$

allow us to expand the states of the first type (labelled by $\tilde{\Lambda}\tilde{M}$) in terms of the states dual to those of the second type (labelled by $\tilde{j}\tilde{m}$) or vice versa. The resubducing coefficients for expansion of the states of a basis C in terms of the states of a basis D may be denoted (see § 5) as the elements of the matrix C^D instead of the brackets $\langle C \,|\, \bar{D} \rangle$, where the dual to D is denoted by $\bar{D}$.

Taking into account that the labels ω in (1.5a), (1.5b) and (1.6) may be replaced by a couple of dual multiplicity labels (a subscript and a supercript) the states of the differential non-canonical bases (labelled by a superscript) may be obtained by means of non-orthonormal RC which satisfy the standard boundary conditions

$$\langle \lambda^{a'} \mu m \,|\, \lambda j m \rangle = \delta_{a', jm} \tag{1.7a}$$

or

$$\langle \Lambda \lambda^{\bar{a}} \mu \,|\, \Lambda M \mu \rangle = \delta_{\bar{a}, \Lambda M} \tag{1.7b}$$

for jm or ΛM which are linearly dependent in the same way as $\tilde{j}\tilde{m}$ or $\tilde{\Lambda}\tilde{M}$ (and which satisfy some additional inequalities when the superfluous states with a subscript appear). Of course, the weight coefficients of the dual bases form mutually inverse matrices.

The dual isofactors and resubducing coefficients were constructed hitherto in different mutually independent ways when the expressions obtained were checked (sometimes after transformation with some triangle matrix inverse to one of the (1.6) type) for satisfaction of the first or second criterion.

Otherwise (cf Ališauskas 1984) the dual coupling coefficients (respectively, isofactors and resubducing coefficients) may be considered as the direct and inverse expansion coefficients. Hecht and Suzuki (1983) obtained, taking into account such considerations, two classes of expressions for some multiplicity-free isofactors of $SU_3 \supset SO_3$. Of course, the dual coupling coefficients coincide numerically in multiplicity-free cases.

Ališauskas (1984)† used the polynomial states of the mixed tensor irreps $(\lambda \dot{0} \mu)$ of U_n restricted to O_n as the generating functions of the bilinear combinations of isofactors of $SU_4 \supset SU_2 \times SU_2$ and $SU_n \supset SO_n$ for coupling $(p\dot{0}) \times (\dot{0}q)$ to $(\lambda \dot{0} \mu)$. An alternative approach—the use of the polynomial direct product states of $(p\dot{0}) \times (\dot{0}q)$ (coupled in frames of SO_n Wigner–Racah algebra) in the role of the generating functions—allowed us to obtain a new expression for the isofactors of dual type, which couple to the stretched (S) states of $SU_n \supset SO_n$ and satisfy the corresponding boundary condition. Although this expression is slightly more complicated than those derived by Ališauskas (1983a) (see § 3 of Ališauskas 1984) it is more symmetric. In addition it is similar to the above mentioned expression of Ališauskas (1984) for bilinear combinations of isofactors.

It turned out that these two expressions for dual isofactors may be obtained (up to an elementary multiplier) each from another by means of special substitution of the non-vanishing parameters (i.e. by means of a discrete analytical continuation procedure).

This new class of relations between the dual isofactor or other resubducing coefficients with analytical multiplicity labels may be associated with the transition to the inverse element of a group in view of the fact that the matrix elements of group generators after these substitutions turn into transposed ones with opposite sign, i.e. into the matrix elements of dual basis (with some renormalisation factors). Also the proper values of the Casimir operators turn into those corresponding to contragredient irreps.

In this way the analytical inversion symmetry of the dual resubducing coefficients or isofactors was introduced (Ališauskas 1986b). It is a generalisation of the mirror reflection symmetry‡ associated with the substitution $j \to -j \to 1$ and considered by Jucys *et al* (1965) and Jucys and Bandzaitis (1965, 1977) in the case of SU_2 and introduced for the non-simple-reducible compact Lie groups by Ališauskas *et al* (1967) and Ališauskas and Jucys (1967). Although it was demonstrated that the substitutions, defined as a mirror reflection, induce the contragrediency transformation without any phase factor, there was no more essential application of the mirror reflection symmetry in the Wigner–Racah calculus of the non-simple-reducible Lie groups. (This statement does not concern the substitution group technique of Ališauskas and Jucys (1967)

† In addition to the corrections given in the corrigendum (1985) of Ališauskas (1984) it is necessary to change -5 to -3 in the exponent of 2 and to include $z!$ in the denominator of (6.6), to exchange Δ_0 and δ_0 in the fourth line from the bottom of p 2912, to replace $W_n(p, l_2)$ by $W_n(p_2, l_2)$ in (6.9), to change the last double factorial [...]!! to a simple one in the numerator of (7.1), to change the sign of l_2' to plus in the exponent of 2 and replace the corresponding factors by $(L_1+L_2+n-4-2t)!!/(L_1+L_2+n-4+2k)!!$ in (7.3). Some other errata are also corrected in Ališauskas (1986a).

‡ This term seems to be inappropriate (contradictory) in view of our investigation, as well as its former interpretations.

used, incidentally, by Ališauskas (1978a, b, 1984).) The difficulties arose because the mirror reflection symmetry was determined only for the canonical basis states of the completely parametrised irreps and the operations with the non-orthogonal multiplicity labels were indefinite.

Otherwise, the analytical inversion symmetry includes the relations between the non-orthonormal resubducing coefficients for the direct and inverse expansion and, thus, the relations between the biorthogonal systems of isofactors or other transformation coefficients as defined by (1.5)-(1.7) and represented in the form of the factorial sums. It should be noted that in the case of the overcomplete bases the analytical inversion, as a rule, gives the expansion in terms of all the states including linearly dependent ones (cf conception of the pseudoisofactors (Ališauskas 1983c, 1984)).

The substitution of analytical inversion for the parameters of irreps λ, μ, j, m, Λ, M are chosen between such compositions of the usual contragrediency transformation with the elements of the substitution group which leave invariant the zero-valued parameters. Let us note that the substitution groups of irreps of the classical Lie groups have been found (cf Ališauskas and Jucys 1967, Ališauskas 1983c) after examination of the invariancy properties of the characters of irreps (cf Weyl 1925, 1926, 1939). They include the permutations of the partial hooks (cf Baird and Biedenharn 1964) for U_n, SO_n and Sp_{2n}, as well as the reflections of the hooks for SO_n and Sp_{2n} (see (9)-(11) of Ališauskas 1983c) together with their compositions. The substitution group is isomorphic to the corresponding Weyl group of the weight space.

The typical applications of the substitution group are being developed for the isofactors and recoupling coefficients (Ališauskas 1978a, b, 1984). Since in these cases for each coupling the one of two irreps to be coupled is fixed, the equivalent substitutions (particularly the hook permutations) should be applied to the remaining two irreps (including the resulting one). Therefore the operations of the substitution groups may be associated with the Weyl operations in the Biedenharn *et al* (1967, 1985) type pattern of canonical tensor operator (corresponding to the fixed irrep). Similarly the substitution $j_1 \rightarrow -j_1 - 1$ is associated with the reflection of k in the Clebsch-Gordan coefficients of SU_2

$$\begin{bmatrix} j_1 & j_2 & j_1+k \\ m_1 & m_2 & m \end{bmatrix}$$

(cf (17.11) of Jucys and Bandzaitis 1977).

The contragredient irreps of U_n are usually obtained after reflection and lexical ordering of the parameters of the Young tableaux, when for SO_{4n+2} only the reflection of the $(2n+1)$th parameter of irreps is needed. In the remaining (SO_n, O_n and Sp_{2n}) cases the contragredient irreps are equivalent to the starting ones.

As a rule the substitution of analytical inversion includes the maximum of reflections and the minimum of permutations of the hooks corresponding to the non-zero-valued parameters of all the irreps. Although the substitutions of analytical inversion for the multiplicity labels $\tilde{j}$, $\tilde{m}$ or $\tilde{\Lambda}$, $\tilde{M}$ resemble those for the remaining parameters, it is generally impossible to relate them to the invariance of the proper values of the Casimir operators corresponding to these labels. (It is obvious because these Casimir operators do not commute with the remaining ones.) The substitution group and the complementary group techniques allowed us to base the behaviour of the multiplicity labels in many cases when the properties of the matrix elements of the group generators may not be considered immediately. Since no one irrep remains untouched it is impossible to use the analytical inversion for the canonical unit tensor operators as defined by Biedenharn *et al* (1967, 1985).

In this paper, the analytical inversion symmetry is demonstrated first of all on special isofactors of $SU_n \supset SO_n$ considered by Mišauskas (1984). This symmetry allowed us to obtain new expressions for the bilinear combinations of isofactors which couple the irreps $(p\dot{0})_n \times (\dot{0}q)_n$ to $(\lambda\dot{0}\mu)_n$ and $(p_1\dot{0})_n \times (p_2\dot{0})_n$ to $(\lambda\nu\dot{0})_n$ as well as for the resubducing coefficients of $Sp_4 \supset SU_2 \times SU_2$ to $Sp_4 \supset U_2$. These new expressions contain the triple sums instead of the fourfold sums presented by Ališauskas (1984).

The new expressions for the isofactors may be used for SU_3, SU_4 and SU_n irreducible decompositions of the nuclear Hamiltonian and the nuclear density matrix as well as for constructing collective basis functions in a two-dimensional case (for references see Ališauskas 1984, 1986a). The whole coupling coefficients of the symmetric irreps of SU_n in $SO_n \supset SO_{n-1} \supset SO_{n-2} \supset \ldots \supset SO_2$ basis may be obtained taking into account also the results by Norvaišas and Ališauskas (1974)† or Ališauskas (1987).

The expansion coefficients of the canonical basis states of SU_3 in terms of the Elliott (1958) states may also be obtained by means of the analytical inversion of the transformation brackets by Asherova and Smirnov (1970). Simultaneously the expansion of the dual to the Elliott basis states in terms of the canonical basis states is found. This result allowed us to obtain an expression for $SU_3 \supset SO_3$ isofactors, which couple two Elliott states to the resulting Elliott state. It should be noted that the known expressions (see, e.g., Asherova and Smirnov (1970) or Castilho-Alcarás and Vanagas (1987)) allow us to expand the direct product of the Elliott states in terms of the dual to the Elliott states or to couple the dual of the Elliott states to the Elliott states (cf Ališauskas 1978a).

2. New solution of the boundary value problem for coupling $(p\dot{0})\times(\dot{0}q)$ to $(\lambda\dot{0}\mu)$ and analytical inversion of $SU_n \supset SO_n$ isofactors

Let us take the same elementary permissible diagrams (EPD) which were used by Ališauskas (1984) for construction of the polynomial states of the mixed tensor irreps $(\lambda\dot{0}\mu)$ of SU_n restricted to SO_n. The direct product state of SU_n for the representation $(p\dot{0})\times(\dot{0}q)$ restricted to the representation $l_1 \times l_2$ of SO_n and coupled in frames of SO_n to the irrep$[L_1L_2]$ may be constructed with the help of the projection operators of the complementary group $Sp(2, R)\times Sp(2, R)$ (see (4.3) of Ališauskas 1986a). For this purpose the normalised monomial

$$N(\eta\xi)^{(l_1+l_2-L_1-L_2)/2}[\eta\xi]^{L_2}_{1,1}\eta_{\mathrm{hw}}^{(l_1-l_2+L_1-L_2)/2}\xi_{\mathrm{h\bar{w}}}^{(l_2-l_1+L_1-L_2)/2} \tag{2.1}$$

(representing a state of the definite irreps $(\lambda'\dot{0}\mu') = (\frac{1}{2}(l_1-l_2+L_1+L_2), \dot{0}, \frac{1}{2}(l_2-l_1+L_1+L_2))$ of SU_n and $[L_1L_2]$ of SO_n) should be acted on with the operators $P^{l_1(n)}_{p,l_1}(\eta)$ and $P^{l_2(n)}_{q,l_2}(\xi)$. An expansion in all EPD introduced by Ališauskas (1984) appears after transferring the annihilation operators through the creation operators. A simple isofactor needs to be used for renormalisation of the first direct product state obtained.

Now the monomials of the type

$$(\eta\eta)^a(\xi\xi)^b[\eta\xi]^{L_2}_{1,1}\eta_{\mathrm{hw}}^{l_1'-L_2}\xi_{\mathrm{hw}}^{l_2'-L_2}$$

† Some errata of Norvaišas and Ališauskas (1974), Ališauskas (1978a, b, 1982b, 1983c) and Ališauskas and Norvaišas (1979, 1980) are corrected in Ališauskas (1986c) (see also Ališauskas and Norvaišas 1980).

may be expanded in terms of the coupled $SU_n \supset SO_n$ states. The expansion coefficients (with the corresponding normalisation factors are the particular cases with $l_1' + l_2' = L_1 + L_2$ of special isofactors considered by Ališauskas (1983a) (see § 3 of Ališauskas 1984) which are multiples of the ${}_3F_2(1)$ series and, therefore, they may be transformed to the more convenient form by the methods of Jucys and Bandzaitis (1965, 1977) used (see §§ 13 and 14) for the Clebsch-Gordan coefficients of SU_2 (cf § 4.3 of Slater 1966). At last, the elementary action with the operator $(\eta\xi)^c$ allows us to represent the general direct product state as an expansion in terms of the coupled polynomial (B) states. In such a way the following new expression for the special isofactors of $SU_n \supset SO_n$ considered in § 3 of Ališauskas (1984) was obtained:

$$
\begin{aligned}
&\begin{bmatrix} (p\dot{0} & (\dot{0}q) & (\lambda\dot{0}\mu)_s \\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2] \end{bmatrix} \\
&= [(p-\lambda)!(\lambda+\mu+n-1)!(\lambda+q+n-1)!(2l_1+n-2)(2l_2+n-2) \\
&\quad \times (2L_1+n-2)!!(2L_2+n-4)!!(L_1+L_2+n-3)!(L_1-l_{10})!(L_1-l_{20})!]^{1/2} \\
&\quad \times \left(\frac{2^{p+q-l_1-l_2+2L_2+n-3}}{(2l_{10}+n-2)!!(2l_{20}+n-2)!!}\right)^{1/2} \frac{\nabla_{n[0,1,2,4]}(l_1l_2;\, L_1L_2)}{(\lambda+\mu+L_1-L_2+n-2)!!} \\
&\quad \times \frac{W_n'(\lambda, l_{10})\, W_n'(\mu, l_{20})}{W_n(p, l_1)\, W_n(q, l_2)} \sum_{z,\alpha,\beta,u} \frac{(-1)^{\psi+\alpha+\beta+z} 2^{\alpha+\beta-u-z}}{z!(\alpha-z)!(\beta-z)!(l_1+l_2-L_1+L_2+n-4-2z)!!} \\
&\quad \times \frac{(2l_1-2\alpha+n-4)!!(2l_2-2\beta+n-4)!![\frac{1}{2}(p-l_1)+\alpha]!}{[\frac{1}{2}(l_1-l_2+L_1-L_2)-\alpha+z]![\frac{1}{2}(l_2-l_1+L_1-L_2)-\beta+z]![\frac{1}{2}(l_1+l_2-L_1-L_2)-\alpha-\beta]!} \\
&\quad \times \frac{[\frac{1}{2}(q-l_2)+\beta]![\frac{1}{2}(l_1-l_2+L_1-L_2)-\alpha+\beta]![\frac{1}{2}(l_2-l_1+L_1-L_2)+\alpha-\beta]!}{[\frac{1}{2}(\lambda+\mu+p+q-l_1-l_2+L_1+L_2)+\alpha+\beta+n-1]!\,u!\,[\frac{1}{2}(p-\lambda-l_1+l_{10})+\alpha+u]!} \\
&\quad \times \frac{(\lambda+\mu+L_1-L_2+n-2+2u)!!}{[\frac{1}{2}(q-l_2-\mu+l_{20})+\beta-u]![\frac{1}{2}(\lambda-p+l_{10}+l_1)-L_2-\alpha+u]!} \\
&\quad \times \{[\tfrac{1}{2}(\mu-q+l_{20}+l_2)-L_2-\beta+u]!\}^{-1}.
\end{aligned} \tag{2.2}
$$

Here $l_{10}+l_{20} = L_1+L_2$ (for $n=3$, $L_2 = 0$ or 1),

$$
\begin{aligned}
&W_n(p, l) = [(p-l)!!(p+l+n-2)!!]^{1/2} \\
&W_n'(p, l) = [(p+l+n-2)!!/(p-l)!!]^{1/2}
\end{aligned} \tag{2.3}
$$

$$
\nabla_{n[i_1,\ldots,i_k]}(l_1l_2;\, L_1L_2) = \left[\prod_{i \neq i_1,\ldots,i_k} A_i \left(\prod_{i=i_1,\ldots,i_k} A_i\right)^{-1}\right]^{1/2} \tag{2.4}
$$

$$
\begin{aligned}
&A_0 = (l_1+l_2+L_1+L_2+2n-6)!! && A_4 = (l_1+l_2+L_1-L_2+n-2)!! \\
&A_1 = (L_1+L_2-l_1+l_2+n-4)!! && A_5 = (L_1-L_2-l_1+l_2)!! \\
&A_2 = (L_1+L_2+l_1-l_2+n-4)!! && A_6 = (L_1-L_2+l_1-l_2)!! \\
&A_3 = (l_1+l_2-L_1+L_2+n-4)!! && A_7 = (l_1+l_2-L_1-L_2)!!.
\end{aligned} \tag{2.5}
$$

In phase factors we use for $n=3$

$$
\begin{aligned}
&\psi = \tfrac{1}{2}(l_1+l_2-L_1-L_2) \qquad \psi' = \tfrac{1}{2}(l_1-l_2-L_1+L_2) \\
&\psi = \psi' = 0 \qquad \text{otherwise.}
\end{aligned} \tag{2.6}
$$

The isofactors (2.2) couple to the states of the stretched (S) basis and are equal to $\delta_{l_2 l_{20}}$ for $p=\lambda$, $q=\mu$, $l_1+l_2=L_1+L_2$. Although equation (2.2) contains a fourfold sum, some particular cases of it are more convenient than the corrected equation (3.4) of Ališauskas (1984) (e.g. for the small values of $p-\lambda$ or L_1-L_2). In addition equation (2.2) resembles (2.8) of Ališauskas (1984). It turned out that both these expressions for dual isofactors satisfy the following relation of analytical continuation:

$$\sum_{\omega}\begin{bmatrix}(\lambda\dot{0}) & (\dot{0}\mu) & (\lambda\dot{0}\mu)\\ l_{10} & l_{20} & \omega[L_1L_2]\end{bmatrix}\begin{bmatrix}(p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)\\ l_1 & l_2 & \omega[L_1L_2]\end{bmatrix}$$
$$=\begin{bmatrix}(p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)_B\\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2]\end{bmatrix}$$
$$=(-1)^{p-\lambda+(L_1+L_2-l_1-l_2)/2}$$
$$\times\left(\frac{(\lambda+\mu+n-1)(2l_{10}+n-2)(2l_{10}+n-4)(2l_{20}+n-2)(2l_{20}+n-4)}{(2L_1+n-2)(2L_2+n-4)(L_1+L_2+n-3)(\lambda+l_{10}+n-2)(\mu+l_{20}+n-2)}\right)^{1/2}$$
$$\times\begin{bmatrix}(-p-n,\dot{0}) & (\dot{0},-q-n) & (-\lambda-n+1,\dot{0},-\mu-n+1)_S\\ -l_1-n+2 & -l_2-n+2 & (-l_{10}-n+3,-l_{20}-n+3)[-L_2-n+3,-L_1-n+3]\end{bmatrix}. \tag{2.7}$$

Here the substitutions of the parameters of irreps correspond to the contragrediency transformation but the multiplicity labels (l_{10}, l_{20}) satisfy another regularity.

The analytical continuation of discrete functions in this paper acquires a wider sense than earlier (in the substitution group or complementary group technique). The separate summation intervals of the expresssions were fixed in former cases of analytical continuation (cf Ališauskas 1984). Here the restrictions of the summation intervals are changing essentially. Some properties of the analytical inversion procedure are discussed in appendix 1.

In order to base the relation of analytical continuation between the direct and the inverse matrices, the properties of the matrix elements of the group generators should be examined. The corresponding matrix elements of SU_4 generators for the Draayer (1970) projected basis are found by Ališauskas and Norvaišas (1979) (equations (2.6) and (2.7)). The linearly dependent states which have appeared should be expanded with the help of (5.1) and (5.7) of Ališauskas and Norvaišas (1979). The explicit expressions for matrix elements in limits of the complete Draayer basis states are rather bulky even for irreps of the type $(\lambda 0 \mu)$. Taking into account the proportionality of the Draayer and stretched states for the irreps $(\lambda 0 \mu)$ (see the factor in front of the sum on the RHS of (3.3) of Ališauskas and Norvaišas (1980)), the matrix elements of the generators in the basis S may be found. These matrix elements turn into transposed ones (with the opposite sign and the corresponding renormalisation factors) after substitutions (2.7) (used together with the correspondence (2.6) and (2.7) of Ališauskas (1984) and $K_s=S$, $K_T=j_{10}-j_{20}$). Then they become the matrix elements of the renormalised dual basis.

In this way relation (2.7) is associated with an outer automorphism of the group which transforms an element of the group into an inverse one. Together with the behaviour of the biorthogonal system of isofactors, this is the reason the new symmetry qualifies as the analytical inversion symmetry.

The substitution group technique allows us to obtain the relations for analytical inversion for isofactors which couple $(p_1\dot{0})\times(p_2\dot{0})$ to $(\lambda\nu\dot{0})$ in the case of the dual bases A and $\bar{A}$ or Q and $\bar{Q}$ as defined by Ališauskas (1984). For demonstration of the usefulness of the analytical inversion symmetry the expression (2.4b) of Ališauskas (1983a) will be needed, which may be written as follows:

$$\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)_s \\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2] \end{bmatrix}$$
$$= (-1)^{(p-\lambda+l_2-l_{20})/2} 2^{(\mu+l_{10}-l_1+l_2+n-3)/2} \bigg(\frac{(2l_1+n-2)(2l_2+n-2)(p-\lambda)!}{(\lambda+q+n-1)!}$$
$$\times \frac{(\lambda+\mu+n-1)!(L_1+L_2+n-3)!(2L_1+n-2)!!(2L_2+n-4)!!(L_1-l_{20})!}{(2l_{10}+n-2)!!(2l_{20}+n-2)!!(l_{20}-L_2)!} \bigg)^{1/2}$$
$$\times \frac{W_n(p,l_1)\,W'_n(q,l_2)\,W'_n(\lambda,l_{10})\,W'_n(\mu,l_{20})}{[\frac{1}{2}(\lambda+\mu+L_1+L_2)+n-3]!\,\nabla_{n[6,7]}(l_1l_2;\,L_1L_2)}$$
$$\times \sum_{u,v,z} \frac{(-1)^{\psi+v+z}[\frac{1}{2}(\mu-l_{20})+u]!\,[\frac{1}{2}(p+L_1+L_2+l_2)+n-3-u]!}{u!\,[\frac{1}{2}(p-l_1)-u]!\,(p+l_1+n-2-2u)!!\,v!\,(p-\lambda-u-v)!}$$
$$\times \frac{[\frac{1}{2}(p-l_1)-u+z]!\,[\frac{1}{2}(\lambda-l_{10})+v]!\,[\frac{1}{2}(L_1-L_2-l_1+l_2)+z]!}{(q+l_2+n-2-2v)!!\,z!\,[\frac{1}{2}(l_1-l_2+L_1-L_2)-z]!\,[\frac{1}{2}(l_{10}-l_{20}-l_1+l_2)+z]!}$$
$$\times \frac{(p-\lambda+l_2+l_{20}+n-4-2u-2v)!!\,(2l_1+n-4-2z)!!\,2^z}{[\frac{1}{2}(p+L_1+L_2+l_2)+n-3-u-v]!\,[\frac{1}{2}(p-\lambda+l_1-l_{10})-z]!}$$
$$\times \frac{[\frac{1}{2}(p+q+L_1+L_2)+n-3-u-v]!}{[\lambda-\frac{1}{2}(p+l_1)+v+z]!}. \tag{2.8}$$

It should be noted that the substitutions (6.1) and (6.2) of Ališauskas (1984) applied to our equations (2.2) or (2.8) allows us to obtain new expressions for the isofactors which couple the states of representation $(p_1\dot{0})\times(p_2\dot{0})$ to antistretched (A) or quasi-stretched (Q) states of the irrep $(\lambda\nu\dot{0})$ of SU_n restricted to SO_n, similarly to § 6 of Ališauskas (1984).

3. New expressions for bilinear combinations of special isofactors of $SU_n \supset SO_n$

The relation (2.7) applied to (2.8) allows us to obtain a new expression for special isofactors of $SU_n \supset SO_n$

$$\begin{bmatrix} (p\dot{0}) & (\dot{0}q) & (\lambda\dot{0}\mu)_B \\ l_1 & l_2 & (l_{10}l_{20})[L_1L_2] \end{bmatrix}$$
$$= (-1)^{(\lambda-l_{10})/2} 2^{-(\mu+l_{10}-l_1+l_2+n-3)/2} \frac{\nabla_{n[6,7]}(l_1l_2;\,L_1L_2)[\frac{1}{2}(\lambda+\mu+L_1+L_2)+n-2]!}{W_n(p,l_1)\,W'_n(q,l_2)\,W'_n(\lambda,l_{10})\,W'_n(\mu,l_{20})}$$
$$\times \bigg(\frac{(\lambda+\mu+n-1)}{(\lambda+\mu+n-2)!}$$
$$\times \frac{(2l_1+n-2)(2l_2+n-2)(\lambda+q+n-1)!(2l_{10}+n-2)!!(2l_{20}+n-2)!!(l_{20}-L_2)!}{(p-\lambda)!(2L_1+n-2)!!(2L_2+n-4)!!(L_1+L_2+n-3)!(L_1-l_{20})!} \bigg)^{1/2}$$
$$\times \sum_{u,v,z} \frac{(-1)^{\psi+u+v}2^z[\frac{1}{2}(p+L_1+L_2+l_2)+n-2+u+v]!}{u!\,[\frac{1}{2}(\mu-l_{20})-u]!\,[\frac{1}{2}(p+L_1+L_2+l_2)+n-2+u]!\,v!\,[\frac{1}{2}(\lambda-l_{10})-v]!}$$

$$
\times \frac{[\frac{1}{2}(p-l_1)+u]!(p+l_1+n-2+2u)!!(p-\lambda+u+v)!}{[\frac{1}{2}(p+q+L_1+L_2)+n-1+u+v]!(p-\lambda+l_2+l_{20}+n-2+2u+2v)!!}
$$
$$
\times \frac{(q+l_2+n-2+2v)!![\frac{1}{2}(L_1-L_2+l_1-l_2)+z]!}{z![\frac{1}{2}(L_1-L_2-l_1+l_2)-z]![\frac{1}{2}(p-l_1)+u-z]![\frac{1}{2}(l_1-l_2-l_{10}+l_{20})+z]!}
$$
$$
\times \frac{[\frac{1}{2}(p-\lambda+l_1-l_{10})+z]!}{[\frac{1}{2}(p+l_1)-\lambda+v+z]!(2l_1+n-2+2z)!!} \tag{3.1}
$$

which is an alternative to (2.8) of Ališauskas (1984) (cf Ališauskas 1986b). Unfortunately, (2.7) applied to (3.4) of Ališauskas (1984) gives a divergent formula, as well as a substitution of analytical inversion

$$
p_1 \to -p_1-n \qquad p_2 \to -p_2-n \qquad \lambda \to -\lambda-2 \qquad \nu \to -\nu-n+1
$$
$$
l_1 \to -l_1-n+2 \qquad l_2 \to -l_2-n+2 \tag{3.2a}
$$
$$
L_1 \to -L_1-n+2 \qquad L_2 \to -L_2-n+4
$$
$$
\underline{l}_1 \to -\underline{l}_1-n+1 \qquad \underline{l}_2 \to -\underline{l}_2-n+3 \tag{3.2b}
$$

applied to (6.4) of Ališauskas (1984). The new expression for special isofactors (cf the corrected equation (7.1) of Ališauskas (1984))

$$
\begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})_{\bar{A}} \\ l_1 & l_2 & (\underline{l}_2)[L_1L_2] \end{bmatrix}
$$
$$
=(-1)^{(p_1-\nu+l_1-\underline{l}_2)/2+\psi'}2^{-(\nu-L_1+L_2-\underline{l}_2+l_1+l_2+n-3)/2}(\lambda+1)
$$
$$
\times\left(\frac{(2l_1+n-2)(2l_2+n-2)}{(p_1-\nu)!(p_2-\nu)!}\right.
$$
$$
\left.\times\frac{\lambda!(2\underline{l}_2+n-2)!!(2L_1+n-4)!!(L_1-L_2)!(\underline{l}_2-L_2)!(L_1+\underline{l}_2+n-3)!}{(2\underline{l}_1+n-4)!!(2L_2+n-4)!!}\right)^{1/2}
$$
$$
\times\frac{\nabla_{n[2,4,5,6]}(l_1l_2;\,L_1L_2)\,W_n(p_1,l_1)\,W'_n(\lambda+\nu,\,L_1-L_2+\underline{l}_2)}{W'_n(p_2,\underline{l}_2)\,W'_n(\nu,\underline{l}_2)[\frac{1}{2}(\lambda+L_1-L_2)]!}
$$
$$
\times\sum_{u,v,z}\frac{(-1)^{v+z}[\frac{1}{2}(p_1+L_1-L_2-l_2)-u]![\frac{1}{2}(\lambda+\nu-L_1+L_2-\underline{l}_2)+v]!}{u![\frac{1}{2}(\nu-\underline{l}_2)-u]![\frac{1}{2}(p_1+L_1-L_2-l_2)-u-v]![\frac{1}{2}(p_1-l_1)-u]!}
$$
$$
\times\frac{(p_2-\nu+u+v)!(p_2+l_2+n-2+2v)!![\frac{1}{2}(p_1-l_1)-u+z]!2^z}{v!(p_1+l_1+n-2-2u)!![\frac{1}{2}(p_2-p_1-L_1+L_2)+u+v]![\frac{1}{2}(l_1+l_2-L_1-L_2)-z]!}
$$
$$
\times\frac{[\frac{1}{2}(p_2-\nu+L_1-L_2-l_1+\underline{l}_2)+z]!(2l_1+n-4-2z)!!}{z![\frac{1}{2}(L_1-L_2-l_1-l_2)+\underline{l}_2+z]![\frac{1}{2}(\lambda+p_2-l_1)+1+v+z]!}
$$
$$
\times\{(p_2-\nu+l_2+\underline{l}_2+n-2+2u+2v)!![\tfrac{1}{2}(L_1+L_2+l_1+l_2)+n-3-z]!\}^{-1} \tag{3.3}
$$

is obtained from (3.1) by use of the substitutions (6.1) and (6.2) of Ališauskas (1984). Otherwise, (3.3) is related by (3.2) to the new expression mentioned in § 2 for the isofactors which couple to the antistretched (A) states. Special isofactors for coupling $(\lambda+\nu,\dot{0})\times(\nu\dot{0})$ to $(\lambda\nu\dot{0})$ with $l_1-l_2=L_1-L_2$ are used as weight coefficients in the sense of (1.5*b*) for the basis $\bar{A}$ with the linearly independent states $\underline{l}_2 \geqslant \nu-L_1+L_2+\delta_0+\Delta_0$. Here $\delta_0=0$ or 1, $\Delta_0=0$ or 1 so that $\lambda+\nu-L_2-\delta_0$ and $\nu-L_2-\Delta_0$ are even.

The substitution (3.2a) together with

$$\begin{aligned} &L_1 \to -L_2 - n + 3 \qquad L_2 \to -L_1 - n + 3 \\ &\bar{l}_1 \to -\bar{l}_1 - n + 3 \qquad \bar{l}_2 \to -\bar{l}_2 - n + 3 \end{aligned} \tag{3.4}$$

applied to the corrected equation (6.9) of Ališauskas (1984) gives an alternative of the corrected equation (7.4) of Ališauskas (1984)

$$\begin{aligned}
&\begin{bmatrix} (p_1\dot{0}) & (p_2\dot{0}) & (\lambda\nu\dot{0})_{\bar{Q}} \\ l_1 & l_2 & (\bar{l}_1\bar{l}_2)[L_1L_2] \end{bmatrix} \\
&\quad = (-1)^{(p_2-\nu-l_2+\bar{l}_2)/2+\psi} 2^{-(p_1+p_2+L_1+L_2+n-3)/2+l_2} \\
&\quad \times \left(\frac{(2l_1+n-2)(2l_2+n-2)\lambda!}{(p_1-\nu)!(p_2-\nu)!(L_1+L_2+n-3)!} \right. \\
&\quad \times \left. \frac{(2\bar{l}_1+n-2)!!(2\bar{l}_2+n-2)!!(\bar{l}_2-L_2)!}{(2L_1+n-2)!!(2L_2+n-4)!!(L_1-\bar{l}_2)!} \right)^{1/2} (\lambda+1) \\
&\quad \times \frac{W_n(p_2, l_2)\nabla_{n[5,7]}(l_1l_2;\, L_1L_2)}{W'_n(p_1, l_1)\, W'_n(\lambda+\nu,\, \bar{l}_1)\, W'_n(\nu, \bar{l}_2)} \\
&\quad \times \sum_{x,y,z} \frac{(-1)^{x+y+z}[\frac{1}{2}(l_1+l_2-L_1-L_2)+x+z]![\frac{1}{2}(p_2-\nu+l_2-\bar{l}_2)+x+z]!}{x!y!z![\frac{1}{2}(p_2-l_2)-x]!(L_1+L_2+l_1-l_2+n-4-2x)!!} \\
&\quad \times \frac{(L_1-L_2-y)!(p_1+L_1+L_2-l_2+n-4-2x-2z)!!}{(2l_2+n-2+2x)!![\frac{1}{2}(p_1-l_1)-z]![\frac{1}{2}(l_2-\bar{l}_2-p_2+\nu)+x]!} \\
&\quad \times \left(\frac{(L_1+L_2-\lambda+n-4)!!}{(L_1+L_2-\lambda+n-4-2z)!!} \right) \frac{[\frac{1}{2}(p_1-\nu+l_1+\bar{l}_2)-L_2-y]!}{(\bar{l}_2-L_2-y)![\frac{1}{2}(L_1-L_2+l_1-l_2)-y]!} \\
&\quad \times \frac{(l_1+l_2+L_1-L_2+n-2+2x-2y)!!}{[\frac{1}{2}(p_2-\nu+l_1-\bar{l}_1)+z]![\frac{1}{2}(\lambda+l_1+l_2)-L_2+x-y+z+1]!} \\
&\quad \times [(l_1+l_2+L_1-L_2+n-2-2y)!!]^{-1}
\end{aligned} \tag{3.5}$$

if $L_1+L_2-\lambda+n-4 \geqslant 0$. Otherwise the quantity in braces should be replaced by

$$(-1)^z(\lambda-L_1-L_2-n+2+2z)!!/(\lambda-L_1-L_2-n+2)!!. \tag{3.6}$$

The weight isofactors for the basis $\bar{Q}$ differ from those for $\bar{A}$ by the condition $\bar{l}_1+\bar{l}_2=L_1+L_2$ with the parameters $\bar{l}_2 \geqslant L_1-\lambda+\Delta_0$ for the linearly independent states of $\bar{Q}$.

It is clear that the old or the new expressions are more or less preferable in different situations because the regions of their polynomial representability are different (cf Castilho-Alcarás and Vanagas 1987).

4. Once more on the expansion of the projected basis of five-dimensional quasispin

Equation (2.7) together with the substitutions (4.1) of Ališauskas (1984) allows us to find the analytical inversion of the projected basis for the reduction $Sp_4 \supset U_2$ introduced by Smirnov and Tolstoy (1973). The parameters used by Ališauskas (1983b, 1984) need to be substituted by

$$K \to -\Lambda - \tfrac{3}{2} \qquad \Lambda \to -K - \tfrac{3}{2} \qquad T \to -T - 1 \qquad M_T \to -M_T$$

$$V \to -V \qquad \alpha \to -\alpha + K - \Lambda \tag{4.1a}$$

$$I \to -I - 1 \qquad J \to -J - 1 \qquad M \to -M \qquad N \to -N. \tag{4.1b}$$

Analytical inversion of (3.2*a*) of Ališauskas (1983b), together with the usual symmetry relations, allowed us to write the following new expression for the resubducing coefficient:

$$\left\langle \begin{matrix} \langle K\Lambda \rangle \\ IMJN \end{matrix} \middle| \begin{matrix} \langle K\Lambda \rangle \\ \alpha;\, VTM_T \end{matrix} \right\rangle$$

$$= \delta_{V,M-N}\delta_{M_T,M+N}(-1)^{K+\Lambda-I-J} \frac{(2T+1)E(K\Lambda\alpha VT)}{(K+\Lambda+T+1)!(2K-2\Lambda-2\alpha)!}$$

$$\times\left(\frac{(2T)!}{(T-M_T)!(T+M_T)!}\right.$$

$$\times\frac{(2I+1)(2J+1)(I-M)!(J-N)!(J+N)!}{(I+M)!(K+\Lambda-I-J)!(K+\Lambda-I+J+1)!(K+\Lambda+I-J+1)!}$$

$$\times\left.\frac{(2K+1)!(2\Lambda)!(2K+2\Lambda+2)!(K-\Lambda+I-J)!}{(K+\Lambda+I+J+2)!(K-\Lambda-I+J)!(I+J-K+\Lambda)!(K-\Lambda+I+J+1)!}\right)^{1/2}$$

$$\times\sum_{x,y,z}\frac{(-1)^{x+y+z}(2I-x)!(K+\Lambda-I+J+x+1)!(K+\Lambda-I-J+x+z)!}{x!(I-M-x)![K-\alpha-I-\frac{1}{2}(T-M_T)+x]!(K+\Lambda-I-N+x+z+1)!}$$

$$\times\frac{[K-\alpha-I+\frac{1}{2}(T-M_T)+x+z]!(K+\Lambda+T+z+1)!(2K-2\Lambda-y)!}{z!(J+N-z)![\Lambda+\alpha-J+\frac{1}{2}(T-M_T)+z]!y!(K-\Lambda+I-J-y)!}$$

$$\times\frac{(I+J-K+\Lambda+y)![K+\alpha-J+\frac{1}{2}(T+M_T)-y]!}{(2\alpha-y)!(I+J-K+\Lambda-x+y)!(2K+T-I-J+x-y+z+1)!} \tag{4.2}$$

where the notation of Ališauskas (1984) is used. Another alternative version of equation (4.2) of Ališauskas (1984) may be obtained by means of the analytical inversion of (3.2*b*) of Ališauskas (1983b). The weight coefficient in this case has the labels of the canonical basis states of $Sp_4 \supset SU_2 \times SU_2$, $\tilde{I} = K - \alpha$, $\tilde{J} = \Lambda + \alpha$, $\tilde{M} = \frac{1}{2}(T+V)$, $\tilde{N} = \frac{1}{2}(T-V)$.

The analytical inversion of the projected $Sp_4 \supset U_2$ states may be confirmed after examination of the matrix elements of group generators (Smirnov and Tolstoy 1973).

5. Analytical inversion for the Elliott $SU_3 \supset SO_3$ states

As was demonstrated by Ališauskas (1978a) the expansion coefficients $\bar{E}^{+G}$ of $SU_3 \supset SO_3$ basis states $\bar{E}^+$ dual to the Elliott states E^+ in terms of the canonical (Gel'fand-Zetlin) basis states coincide with the expansion coefficients G^{E^+} of the canonical (G) states in terms of the Elliott states:

$$\bar{E}^{+(\lambda\mu,M)G}_{KL;ZI} = \left\langle \begin{matrix} (\lambda\mu)_{\bar{E}^+} \\ KLM \end{matrix} \middle| \begin{matrix} (\lambda\mu) \\ ZI\frac{1}{2}M \end{matrix} \right\rangle = G^{(\lambda\mu,M)E^+}_{ZI;KL}. \tag{5.1}$$

The $SU_3 \supset U_2$ states here are labelled by the parameters

$$Z = \tfrac{1}{3}(\mu - \lambda) - \tfrac{1}{2}Y = m_{23} - \tfrac{1}{2}(m_{12} + m_{22})$$
$$I = \tfrac{1}{2}(m_{12} - m_{22}) \qquad \tfrac{1}{2}M = m_{11} - \tfrac{1}{2}(m_{12} + m_{22}) \tag{5.2}$$

(m_{ik} are the Gel'fand-Zetlin parameters). The Elliott basis states E^+ and E^- are defined as

$$\left|\begin{matrix}(\lambda\mu)_{E^+}\\ KLM\end{matrix}\right\rangle = P^L_{MK}\left|\begin{matrix}(\lambda\mu)\\ \frac{1}{2}\mu\frac{1}{2}\mu\frac{1}{2}K\end{matrix}\right\rangle \tag{5.3a}$$

$$\left|\begin{matrix}(\lambda\mu)_{E^-}\\ KLM\end{matrix}\right\rangle = P^L_{MK}\left|\begin{matrix}(\lambda\mu)\\ -\frac{1}{2}\lambda\frac{1}{2}\lambda\frac{1}{2}K\end{matrix}\right\rangle \tag{5.3b}$$

where the extremal states of the canonical basis are taken on the RHS†. The overlaps of the E^+ states are given in the simplest form as (3.9)‡ of Ališauskas (1982b).

The generators of rank 2 of SU_3 act on the Elliott states E^+ as follows (see (54) of Ališauskas (1983c), cf Elliott (1958), Asherova and Smirnov (1973), De Meyer *et al* (1985)):

$$Q_m\left|\begin{matrix}(\lambda\mu)_{E^+}\\ KLM\end{matrix}\right\rangle = \sum_{L',K',m',M'} \frac{(2L+1)}{(2L'+1)}\begin{bmatrix}L & 2 & L'\\ M & m & M'\end{bmatrix}$$
$$\times\begin{bmatrix}L & 2 & L'\\ K & m' & K'\end{bmatrix} b^{(\lambda\mu)}_{LK';L'K'}\left|\begin{matrix}(\lambda\mu)_{E^+}\\ K'L'M'\end{matrix}\right\rangle \tag{5.4}$$

where

$$b^{(\lambda\mu)}_{LK';L'K'} = \delta_{KK'}(1/\sqrt{6})[2\lambda + \mu + 3 + \tfrac{1}{2}L'(L'+1) - \tfrac{1}{2}L(L+1)]$$
$$- \delta_{K',K+2}\tfrac{1}{4}[(\mu - K)(\mu + K + 2)]^{1/2} - \delta_{K',K-2}\tfrac{1}{4}[(\mu + K)(\mu - K + 2)]^{1/2}. \tag{5.5}$$

The states of the dual basis $\bar{E}^+$ may be defined by the boundary condition of the resubducing coefficients $\bar{E}^{+G}$:

$$\bar{E}^{+(\lambda\mu,M)G}_{KL',\frac{1}{2}\mu\frac{1}{2}\mu} = \delta_{MK} \tag{5.6}$$

for $K \geq \max(L - \lambda + \delta, \Delta + \delta)$, $M \geq \max(L - \lambda + \delta, \Delta + \delta)$, where $\delta = 0$ or 1, $\Delta = 0$ or 1 so that $\lambda + \mu - L - \delta$ and $\lambda - L - \Delta$ are even integers. According to (4.2) and (3.9) of Ališauskas (1978a), the $\bar{E}^+$ states may be expanded in terms of the Bargmann and Moshinsky (1960, 1961) states which in their turn may be expanded in terms of the canonical basis states. Otherwise, the canonical basis states may be expanded in terms of stretched (S) states and the latter in terms of the Elliott states (see (4.22) and (4.14) of Moshinsky *et al* (1975)). Thus, the expression for the resubducing coefficient (5.1) obtained in this way contains a sixfold sum.

It is easy to check that the matrix elements (5.4) after the substitution

$$\lambda \to -\lambda - 2 \qquad \mu \to -\mu - 2 \qquad L \to -L - 1 \qquad M \to -M \qquad K \to -K \tag{5.7}$$

† A new construction for the projected (E) bases of $SU_n \supset SO_n$ in the case of two-parametric irreps (cf Ališauskas 1982b, 1984) is proposed by Petrauskas and Ališauskas (1987). For the bases E, B, $\bar{A}$ and $\bar{Q}$, special isofactors now may be used as the weight coefficients instead of special RC of the chains $SU_3 \supset SO_3 \supset SO_2$ and $SU_3 \supset U_2 \supset SO_2$ of our (E) case.

‡ One of two repeating factors on its RHS should be omitted.

turn into the transposed ones (with a simple multiplier), i.e. into the matrix elements in dual basis. This fact, together with some additional reasoning, allowed us to suppose the following relation of analytical inversion (cfAlišauskas 1986b):

$$G^{(\lambda\mu,M)E^+}_{ZI;KL} = 2[(\lambda+1)(\lambda+\mu+2)]^{1/2}(2L+1)^{-1}E^{+(-\lambda-2,-\mu-2;-M)G}_{-K,-L-1;-Z,-I-1}. \tag{5.8}$$

The expansion coefficients $E^{+(\lambda\mu,M)G}_{KL;ZI}$ of the Elliott states in terms of the canonical basis states were found by Asherova and Smirnov (1970) (see (2.5) ofAlišauskas (1978))†. Analytical continuation of this formula according to (5.8) gives the expansion of the canonical basis states in terms of all the E^+ states of the overcomplete basis, the linearly dependent states being included. The latter may be expanded in terms of the independent states with the help of the equation

$$\left|\begin{matrix}(\lambda\mu)_{E^+}\\ K'LM\end{matrix}\right\rangle = \sum_K R^{(\lambda\mu,L)}_{K',K}\left|\begin{matrix}(\lambda\mu)_{E^+}\\ KLM\end{matrix}\right\rangle \tag{5.9}$$

where

$$\begin{aligned}R^{(\lambda\mu,L)}_{K',K} &= [\delta_{KK'}+(-1)^{\lambda-L}\delta_{K,-K'}](1+\delta_{K'0})^{-1}\\ &\quad\times\left(\frac{(\mu+K')!!(\mu-K')!!(L+K')!(L-K')!}{(\mu+K)!!(\mu-K)!!(L+K)!(L-K)!}\right)^{1/2}\frac{(-1)^{(L-\lambda+\delta-K')/2}}{(K-K')(K+K')}\\ &\quad\times\frac{2K'^{\delta}K^{1-\delta}(L-\lambda+K+\delta-2)!!}{(L-\lambda+\delta-K'-2)!!(L-\lambda+\delta+K'-2)!!(\lambda-L+K-\delta)!!}\end{aligned} \tag{5.10}$$

$K \geqslant \max(\Delta+\delta, L-\lambda+\delta)$, $K' \geqslant \Delta+\delta$ (cf the corrected equations (4.4) and (2.8) ofAlišauskas (1978)).

Thus the following expression for the resubducing coefficients was obtained:

$$\begin{aligned}G^{(\lambda\mu,M)E^+}_{ZI;KL} &= \sum_{K'} R^{(\lambda\mu L)}_{K'K}(-1)^{(M+\mu+K')/2-L-Z}\\ &\quad\times\left(\frac{(\lambda+Z-I)!(\lambda+Z+I+1)!(I+\frac{1}{2}M)!(\mu+K')!!(L-M)!}{2^{2L-M}\lambda!(\lambda+\mu+1)!(I-\frac{1}{2}M)!(\mu-K')!!(L+M)!}\right.\\ &\quad\left.\times\frac{(L-K')!}{(L+K')!}\right)^{1/2}\sum_{z,x,y}\frac{(-1)^{z+x+y}2^{z-x-y}(2L-z)!}{z!(L-M-z)!(L-K'-z)!x!y!}\\ &\quad\times\frac{[\frac{1}{2}(K'+\mu)+\lambda-L+z-y+1]!}{[\frac{1}{2}(K'+M+\mu)-L-Z+z-x-y]!(\lambda+Z-\frac{1}{2}M+x+1)!}\\ &\quad\times\{[\tfrac{1}{2}(\mu+M-K')-Z+y-x]![\tfrac{1}{2}(\mu-M+K')-Z+x-y]!\}^{1/2}\\ &\quad\times\left(\frac{[\frac{1}{2}(\mu-K')+y]!(I-\frac{1}{2}M+x)!}{[\frac{1}{2}(\mu+K')-y]!(I+\frac{1}{2}M-x)!}\right)^{1/2}\\ &\quad\times\begin{bmatrix}\frac{1}{2}\mu & \frac{1}{2}\mu-Z & I\\ \frac{1}{2}K'-y & \frac{1}{2}(M-K')-x+y & \frac{1}{2}M-x\end{bmatrix}.\end{aligned} \tag{5.11}$$

On the RHS the Clebsch-Gordan coefficients of SU_2 appeared after the application of (17.1) of Jucys and Bandzaitis (1977) to (2.5) ofAlišauskas (1978a) in which the Clebsch-Gordan coefficients have been extracted as well.

† The expression with a more convenient region of polynomial representability is given in Castilho-Alcarás and Vanagas (1987).

Of course, the above mentioned combination of (4.2) and (3.9) of Ališauskas (1978a) may be more convenient than (5.11) for special values of parameters (e.g. for small values of λ or μ). However, (5.11) simplifies itself considerably for Z and I close to $\mu/2$, e.g. on the RHS the terms with the parameters

$$|M-K'|>\mu-2Z \tag{5.12}$$

vanish. From (5.12) the additional selection rules for the coupling coefficients of $SU_3 \supset SO_3$ may be obtained (see appendix 2).

The overlap of the $\bar{E}^+$ states (which coincide with the metric tensors of the E^+ states) may be found with the help of equations (4.2) and (5.6) of Ališauskas (1978a) (or with the help of (4.14) of Moshinsky *et al* (1975) and (A3.5) of Ališauskas (1984)). The analytical inversion of the overlaps of the E^+ states found by Asherova and Smirnov (1970) (cf Ališauskas 1978a) allowed us to represent the overlaps of $\bar{E}^+$ states in the following form:

$$\left\langle \begin{matrix}(\lambda\mu)_{\bar{E}^+}\\ K_1LM\end{matrix}\middle| \begin{matrix}(\lambda\mu)_{\bar{E}^+}\\ K_2LM\end{matrix}\right\rangle$$

$$\begin{aligned} &= (\lambda+1)(\lambda+\mu+2)(2L+1)^{-1}\sum_{K_1'K_2'} R^{(\lambda\mu L)}_{K_1'K_1} R^{(\lambda\mu L)}_{K_2'K_2} \\ &\quad\times\left(\frac{(L-K_1')!(L-K_2')![\frac{1}{2}(\mu+K_1')]![\frac{1}{2}(\mu+K_2')]!}{(L+K_1')!(L+K_2')![\frac{1}{2}(\mu-K_1')]![\frac{1}{2}(\mu-K_2')]!}\right)^{1/2} \\ &\quad\times\sum_{z,x}\frac{2^{K_2'-L+z-2x}(2L-z)!(L-K_2'-z+2x)![\frac{1}{2}(\mu-K_2')+x]!}{z!(L-K_1'-z)!(L-K_2'-z)!x![\frac{1}{2}(\mu+K_2')-x]!} \\ &\quad\times\frac{[\lambda-L+\frac{1}{2}(\mu+K_2')-x+z+1]!}{[\frac{1}{2}(K_1'-K_2')+x]![\lambda+\frac{1}{2}(\mu-K_2')+x+2]!}. \end{aligned} \tag{5.13}$$

This expression is convenient for small values of μ and L.

The symmetry properties of the resubducing coefficients (see Ališauskas 1978a) allows us to find $G^{(\lambda\mu,M)E^-}_{ZI;KL}$ as well.

It should be noted that (5.11) together with (4.1) of Ališauskas (1978a) or (4.13) of Moshinsky *et al* (1975) give a new expansion for the Bargmann-Moshinsky states in terms of the canonical states.

6. Conclusion

We have demonstrated rather unexpected properties of the biorthogonal systems of isofactors and resubducing coefficients. The coefficients of the direct and the inverse expansion appeared to be related by a special analytical continuation procedure.

In particular we have demonstrated that the majority of the isoscalar factors of $SU_n \supset SO_n$ and $SU_4 \supset SU_2 \times SU_2$, considered by Ališauskas (1984), as well as the resubducing coefficients of $Sp_4 \supset U_2$ to $Sp_4 \supset SU_2 \times SU_2$, were some incarnations of a certain discrete function, assuming at least three main forms. We succeeded in inversing the expansion of the projected Elliott states of $SU_3 \supset SO_3$ in terms of the canonical basis states and in such a way as to extend the possibilities for expansion in terms of each other of the different bases of SU_3 in addition to those presented by Moshinsky *et al* (1975) and Ališauskas (1978a).

More trivial examples of the analytical inversion symmetry may be found among the elementary resubducing coefficients of $SU_3 \supset SO_3$ (see (4.1) and (4.2) or (4.9) and

(4.10) of Ališauskas (1978a)) which form the triangular matrices. This symmetry allows us to obtain new expressions for the multiplicity-free isofactors (e.g. the expression (9) and (12)-(15) of Ališauskas *et al* (1972) for special isofactors of $U_n \supset U_{n-1}$ may be related in terms of the analytical inversion symmetry).

The possibility of obtaining new expressions for isofactors and other transformation coefficients is very important in connection with the polynomial representability problem (see, e.g., Castilho-Alcarás and Vanagas 1987).

We have not exhausted here all the examples of the non-multiplicity-free isofactors for which the new symmetry may be useful. The $SU_n \supset SO_n$ projected basis case is considered by Petrauskas and Ališauskas (1987). Some new results are also obtained for the $SU_3 \times SU_3 \supset SU_3$ case, i.e. for the isofactors of the canonical basis of SU_3. The reasons for which the analytical inversion symmetry may be ineffective are discussed in appendix 1.

Appendix 1. Discussion of analytical inversion procedure

Let us discuss some common properties of the discrete functions of the Wigner-Racah calculus (isofactors, recoupling or resubducing coefficients or, briefly, WR functions) which are important for the analytical inversion procedure as well as for other analytical continuation techniques. The WR function for analytical inversion should be represented as factorial sums, each separate sum being equivalent, as a rule, to some ${}_{k+1}F_k(1)$ series (with the only known exceptions appearing in Asherova and Smirnov (1970) and § 5). For every WR function the arguments of the simple factorials (or the halves of the arguments of double factorials) under the square root sign are especially significant parameters because the identical linear combinations of parameters are usually simply reproducible from the arguments of factorials depending on the summation parameters. Therefore, these arguments are useful when controlling the explicit expressions (e.g. the factor $[\frac{1}{2}(\nu - L_2) - x + y]!$ in (7.1) of Ališauskas (1984) should be corrected to $[\frac{1}{2}(\nu - \underline{l}_2) - x + y]!$ because the factor $(\nu - \underline{l}_2)!!$ appears under the sign $\surd$, and $\frac{1}{2}(\nu - \underline{l}_2)$ is the sum of z, $x - y - z$ and $\frac{1}{2}(\nu - \underline{l}_2) - x + y$.

Naturally, the large majority of factorials under the sign $\surd$ pass from the numerator into the denominator and vice versa for dual WR functions. The renormalisation factor is needed for compensation of disturbances from this rule, appearing after analytical inversion and formal use of the relation

$$(-x-1)!/(-y-1)! = (-1)^{y-x} y!/x! \tag{A1.1}$$

(cf Jucys and Bandzaitis 1965, 1977).

Now let us select such linear combinations of parameters from those above mentioned arguments of factorials that the WR function vanishes for any of them taking negative integer values, in a similar way as the Clebsch-Gordan or $6j$-coefficients of SU_2 do for the negative integer values of $j \pm m$ or $j_1 + j_2 - j_3$ (see Regge 1958, Bargmann 1962, Shelepin 1964). These linear combinations correspond to some branching or selection rules and will be called the main Regge-Bargmann-Shelepin type parameters (RBSP). For example, the following main RBSP

$$\begin{array}{lllll} p - \lambda = q - \mu & \frac{1}{2}(p - l_1) & \frac{1}{2}(q - l_2) & \frac{1}{2}(L_1 - L_2 - l_1 + l_2) & \\ \frac{1}{2}(L_1 - L_2 + l_1 - l_2) & \frac{1}{2}(l_1 + l_2 - L_1 - L_2) & \frac{1}{2}(\lambda - l_{10}) & \frac{1}{2}(\mu - l_{20}) & \\ L_1 - l_{20} = l_{10} - L_2 & L_1 - l_{10} = l_{20} - L_2 & & & \end{array} \tag{A1.2}$$

are chosen for the dual isofactors represented by (2.2), (2.8) and (3.1).

Since the main RBSP do not depend on n this parametrisation may be incomplete. For example, the set (A1.2) may be replenished by the following additional RBSP:

$$\tfrac{1}{2}(L_1+L_2-l_1+l_2+n-4) \qquad \tfrac{1}{2}(L_1+L_2+l_1-l_2+n-4)$$
$$\tfrac{1}{2}(l_1+l_2-L_1+L_2+n-4) \tag{A1.3}$$

which turn into the homogeneous linear combinations of the usual parameters of the WR function for a definite n, similar to the main RBSP. It should be noted that the additional RBSP cannot be negative for positive values of all the main RBSP parameters. The analytical continuation of the WR function with the negative integer values of the several (not single) main RBSP may be non-vanishing. Otherwise, it should be noted that the WR functions satisfying special boundary conditions vanish for negative values of some bilinear combinations which do not belong to the RBSP introduced. (We are not discussing here the zeros of the WR functions in the region allowed by branching and selection rules.)

In the explicit expressions of the WR functions the RBSP may play several different roles.

(a) The RBSP is equal to the lengths of an interval for a summation parameter (or parameters) and appears in the numerator under the $\sqrt{}$ sign.

(b) The RBSP appears in the denominator under the $\sqrt{}$ sign and may be reproduced (re-expressed) similarly as $a-b$ from $(a+z)!/(b+z)!$ or $(a-z)!(b-z)!$, where z is a summation parameter. (The elimination of the corresponding factors for $a=b$ is permitted solely if the summation interval remains unchanged.)

(c) The RBSP may be re-expressed similarly as $s-a$ from $(a-z)!/(s+1-z)!$ and appears in the numerator under the $\sqrt{}$ sign. The WR function vanishes because the special summation formula is valid.

(d) The RSBP corresponds to a displacement from Saalschutzian type series (see Slater 1966) and may be re-expressed in a more complicated way in comparison with other cases.

The set and the number of the RBSP of type (a) determine the convenience of any expression for the WR function and the region of its polynomial representability. The analytical inversion procedure exchanges as a rule the roles of RBSP of types (a) and (b). Therefore, the appearance of RBSP of type (b) for each summation parameter may be essential for the existence of the analytical inversion symmetry (e.g. this is the reason why expression (2.6) of Ališauskas (1978a) may be not inverted).

In order to escape the false restrictions of the summation parameters (and associated with them the multivaluedness of the functions obtained) it is necessary to be cautious to some extent with the analytical inversion of the subfunctions included in the structure of the WR function in a similar way as the Clebsch-Gordan coefficients of SU_2 are in (5.11). The region of non-vanishing analytical continuation of the subfunction multiplied by a factor under the $\sqrt{}$ sign may be wider as the corresponding region for the analytical continuation of the subfunction alone. Such a situation appears, e.g., with the Clebsch-Gordan coefficient of SU_2 in the case of analytical inversion of (5.2) of Ališauskas (1984) (see Petrauskas and Ališauskas 1987).

The following typical difficulties of the analytical inversion should be mentioned.

(i) When the WR function is not a completely analytical function because some of its parameters accept only a finite set of values. (This note does not concern the expressions for isofactors of $SU_3 \supset SO_3$ with special values of the parameter $L_2=0$ or 1 in the general $SU_n \supset SO_n$ situation.)

(ii) When the WR function depends on two sets of multiplicity labels of irreps, for example, the overlap of some non-orthogonal basis when it is expressed as a particular case of the corresponding RC without decreasing the number of sums or the overlap of the different bases with decreasing cardinality of the common region of non-overcompleteness.

The cardinality of the region of exact completeness seems to be important for the analytical inversion in general. Since the analytical inversion of the known explicit overlaps $\langle \bar{A} | \bar{A} \rangle$, $\langle \bar{Q} | \bar{Q} \rangle$, $\langle E | E \rangle$ for $SU_n \supset SO_n$ $(n \geqslant 4)$ was unsuccessful, as well as the inversion of $\langle \bar{A} | E \rangle$, only the analytical inversion of $\langle \bar{Q} | E \rangle$ together with some methods used to prove (5.4) and (7.3) of Ališauskas (1984) allowed us (Petrauskas and Ališauskas 1987) to construct the overlaps of the dual bases $\langle \bar{E} | \bar{E} \rangle$, $\langle Q | Q \rangle$ and $\langle A | A \rangle$ for the two-parametric irreps of $SU_n \supset SO_n$ $(n \geqslant 3)$.

Appendix 2. On the coupling coefficients for the Elliott states

A well known construction (Engeland 1965, Vergados 1968, Asherova and Smirnov 1970) allowed us to write an expression for isofactors which couple the Elliott states of two irreps $(\lambda_n \mu_1)$ and $(\lambda_2 \mu_2)$ to the Elliott states of the representation $(\lambda \mu)$:

$$\begin{bmatrix} (\lambda_1 \mu_1)_{\bar{E}^+} & (\lambda_2 \mu_2)_{\bar{E}^+} & {}^{\rho}(\lambda \mu)_{E^+} \\ K_1 L_1 & K_2 L_2 & KL \end{bmatrix}$$

$$= \sum_{Z_1, Z_2, I_1, I_2, M_1, M_2} \begin{bmatrix} L_1 & L_2 & L \\ M_1 & M_2 & K \end{bmatrix} G^{(\lambda_1 \mu_1, M_1) E^+}_{Z_1 I_1; K_1 L_1} G^{(\lambda_2 \mu_2, M_2) E^+}_{Z_2 I_2; K_2 L_2}$$

$$\times \begin{bmatrix} (\lambda_1 \mu_1) & (\lambda_2 \mu_2) & {}^{\rho}(\lambda \mu) \\ Z_1 I_1 & Z_2 I_2 & \frac{1}{2}\mu \frac{1}{2}\mu \end{bmatrix} \begin{bmatrix} I_1 & I_2 & \frac{1}{2}\mu \\ \frac{1}{2}M_1 & \frac{1}{2}M_2 & \frac{1}{2}K \end{bmatrix} \tag{A2.1}$$

where $Z_1 + Z_2 = \frac{1}{2}\mu - s$, $s = \frac{1}{3}(\lambda_1 - \mu_1 + \lambda_2 - \mu_2 - \lambda + \mu)$. On the RHS the Clebsch-Gordan coefficients of $SU_2(SO_3)$ appeared, as well as the special isofactors of $SU_3 \supset U_2$. Some selection rules for the parameters K_1, K_2 and K of (A2.1) are caused by the vanishing of the terms with the parameters

$$|K_1' + K_2' - K| > \mu_1 + \mu_2 - \mu + 2s = \tfrac{1}{3}(2\lambda_1 + \mu_1 + 2\lambda_2 + \mu_2 - 2\lambda - \mu) \tag{A2.2}$$

where the summation parameters K_1', K_2' appear in the resubducing coefficients G^{E^+} on the RHS of (A2.1).

Special isofactors of $SU_3 \supset U_2$ on the RHS (A2.1) take the form (cf Ališauskas 1978b, 1982a)

$$\begin{bmatrix} (\lambda_1 \mu_1) & (\lambda_2 \mu_2) & {}^{+, \bar{I}_{21}, +}(\lambda \mu) \\ Z_1 I_1 & Z_2 I_2 & \frac{1}{2}\mu \frac{1}{2}\mu \end{bmatrix}$$

$$= \sum_{i_2} \Bigg[\delta_{i_2 \bar{I}_2} + \Bigg(\frac{(2\bar{I}_2 + 1)[\frac{1}{2}(\mu_1 + \mu) - i_2]! [\frac{1}{2}(\mu_1 + \mu) + i_2 + 1]!}{[\frac{1}{2}(\mu_1 + \mu) - \bar{I}_2]! [\frac{1}{2}(\mu_1 + \mu) + \bar{I}_2 + 1]!}$$

$$\times \frac{(2i_2 + 1)(\mu_2 - \underline{Z}_2 - i_2)! (\mu_2 - \underline{Z}_2 + i_2 + 1)! (\lambda_2 + \underline{Z}_2 - i_2)! (\lambda_2 + \underline{Z}_2 + i_2 + 1)!}{(\mu_2 - \underline{Z}_2 - \bar{I}_2)! (\mu_2 - \underline{Z}_2 + \bar{I}_2 + 1)! (\lambda_2 + \underline{Z}_2 - \bar{I}_2)! (\lambda_2 + \underline{Z}_2 + \bar{I}_2 + 1)!} \Bigg)^{1/2}$$

$$\times \left(\frac{(\bar{I}_2 + \underline{Z}_2)! (i_2 - \underline{Z}_2)! [i_2 + \frac{1}{2}(\mu - \mu_1)]! [\bar{I}_2 + \frac{1}{2}(\mu_1 - \mu)]!}{(i_2 + \underline{Z}_2)! (\bar{I}_2 - \underline{Z}_2)! [\bar{I}_2 + \frac{1}{2}(\mu - \mu_1)]! [i_2 + \frac{1}{2}(\mu_1 - \mu)]!} \right)^{\frac{1}{2}\mathrm{sign}\, s}$$

$$\times\frac{(-1)^{\bar{I}_2-B}(\bar{I}_2+B)!}{(\bar{I}_2-i_2)(\bar{I}_2+i_2+1)(\bar{I}_2-B)!(i_2+B)!(B-i_2-1)!}\Bigg]$$

$$\times\frac{[(2i_2+1)(2I_1+1)(2I_2+1)(\lambda_1+Z_1-I_1)!(\lambda_1+Z_1+I_1+1)!]^{1/2}}{[\lambda_1!(\lambda_1+\mu_1+1)!]^{1/2}\nabla(\frac{1}{2}\mu_1-Z_1,i_2,I_2)}$$

$$\times(-1)^{(\mu-\mu_1)/2+i_2-2I_1}\frac{\Gamma(\lambda_2\mu_2I_2Z_2)}{\Gamma(\lambda_2\mu_2i_2Z_2)}\begin{Bmatrix} I_1 & I_2 & \frac{1}{2}\mu \\ i_2 & \frac{1}{2}\mu_1 & \frac{1}{2}\mu_1-Z_1 \end{Bmatrix} \tag{A2.3}$$

if the outer multiplicity labels of Ališauskas (1978b) are used. Here

$$B=\tfrac{1}{2}(\mu_2-\lambda_1+\lambda_2-\lambda+|s|) \tag{A2.4}$$

$$Z_2=\tfrac{1}{2}(\mu-\mu_1)-s$$

$$\nabla(abc)=\left(\frac{(a-b+c)!(a+b-c)!(a+b+c+1)!}{(b+c-a)!}\right)^{1/2} \tag{A2.5}$$

$$\Gamma(\lambda\mu IZ)=\left(\frac{(I+Z)!(\lambda+Z-I)!(\lambda+Z+I+1)!}{(I-Z)!(\mu-Z-I)!(\mu-Z+I+1)!}\right)^{1/2}. \tag{A2.6}$$

The factor $(\ldots)^{\frac{1}{2}\operatorname{sign}s}=1$ for $s=0$. The sum over i_2 reduces to single terms if

$$2\lambda_1+\mu_1-\lambda_2-2\mu_2+\lambda-\mu\geqslant 0 \tag{A2.7a}$$

or

$$\lambda_1-\mu_1-2\lambda_2-\mu_2+2\lambda+\mu\geqslant 0. \tag{A2.7b}$$

The isofactors (A2.3) satisfy the boundary condition for $Z_1=I_1=\frac{1}{2}\mu_1$, $I_2\geqslant B$. Therefore, they are very convenient for use in the Wigner-Eckart theorem. In this situation the corresponding boundary values of the matrix elements in the canonical SU_3 basis may be used as the reduced matrix elements.

The overlaps and the metric tensors (which form the inverse matrices of the overlap matrices) for these systems of isofactors are also given by Ališauskas (1978b, 1982a) in the form of double sums (see also Ališauskas 1983c) and will be discussed in our next paper.

It is not difficult to write an expression for the isofactors which reduce the direct product of two Elliott states in terms of the Elliott states of the resulting irrep (see (6.5) of Ališauskas 1978a). It should be noted that the latter quantities do not coincide with (A2.1). The selection rules for the parameters K_1, K_2 and K are less effective in this case, as well as for variants of (A2.1) with a $\bar{E}^+$ basis particularly replaced by a $\bar{E}^-$ basis.

Similar selection rules of the multiplicity labels appear for the isofactors of $SU_3\supset SO_3$ which allow us to expand the direct product of the Bargmann-Moshinsky states in terms of the same states (cf Ališauskas and Norvaišas 1985).

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